

# ON INDEX-EXPONENT RELATIONS OVER HENSELIAN FIELDS WITH FINITE OR LOCAL RESIDUE FIELDS

I.D. CHIPCHAKOV

**ABSTRACT.** Let  $(K, v)$  be a Henselian valued field with a residue field  $\hat{K}$ , and let  $p$  be a prime number. This paper determines the Brauer  $p$ -dimension of  $K$ , provided that  $p \neq \text{char}(\hat{K})$  and  $\hat{K}$  is a  $p$ -quasilocal field which is properly included in its maximal  $p$ -extension. When  $\hat{K}$  is a local field with  $\text{char}(\hat{K}) \neq p$ , it fully describes index-exponent relations in the  $p$ -component of the Brauer group  $\text{Br}(K)$ . The same goal is achieved in case  $(K, v)$  is maximally complete,  $\text{char}(K) = p$  and  $\hat{K}$  is a local field.

## 1. Introduction

Let  $E$  be a field,  $\mathbb{P}$  the set of prime numbers, and for each  $p \in \mathbb{P}$ , let  $E(p)$  be the maximal  $p$ -extension of  $E$  in a separable closure  $E_{\text{sep}}$ , and  $r_p(E)$  the rank of the Galois group  $\mathcal{G}(E(p)/E)$  as a pro- $p$ -group (put  $r_p(E) = 0$ , if  $E(p) = E$ ). Denote by  $s(E)$  the class of finite-dimensional associative central simple  $E$ -algebras, and by  $d(E)$  the subclass of division algebras  $D \in s(E)$ . For each  $A \in s(E)$ , let  $[A]$  be the equivalence class of  $A$  in the Brauer group  $\text{Br}(E)$ , and  $D_A$  a representative of  $[A]$  lying in  $d(E)$ . The existence of  $D_A$  and its uniqueness, up-to an  $E$ -isomorphism, is established by Wedderburn's structure theorem (cf. [29], Sect. 3.5), which implies the dimension  $[A: E]$  is a square of a positive integer  $\deg(A)$  (the degree of  $A$ ). It is known that  $\text{Br}(E)$  is an abelian torsion group, so it decomposes into the direct sum of its  $p$ -components  $\text{Br}(E)_p$ , taken over  $\mathbb{P}$  (see [29], Sects. 3.5 and 14.4). The Schur index  $\text{ind}(D) = \deg(D_A)$  and the exponent  $\exp(A)$ , i.e. the order of  $[A]$  in  $\text{Br}(E)$ , are invariants of both  $D_A$  and  $[A]$ . Their general relations and behaviour under scalar extensions of finite degrees are described as follows (cf. [29], Sects. 13.4, 14.4 and 15.2):

- (1.1) (a)  $\exp(A) \mid \text{ind}(A)$  and  $p \mid \exp(A)$ , for each  $p \in \mathbb{P}$  dividing  $\text{ind}(A)$ . For any  $B \in s(E)$  with  $\text{ind}(B)$  prime to  $\text{ind}(A)$ ,  $\text{ind}(A \otimes_E B) = \text{ind}(A) \cdot \text{ind}(B)$ ; if  $A, B \in d(E)$ , then the tensor product  $A \otimes_E B$  lies in  $d(E)$ ;  
 (b)  $\text{ind}(A)$  and  $\text{ind}(A \otimes_E R)$  divide  $\text{ind}(A \otimes_E R)[R: E]$  and  $\text{ind}(A)$ , respectively, for each finite field extension  $R/E$  of degree  $[R: E]$ .

As shown by Brauer (see, e.g., [29], Sect. 19.6), (1.1) (a) determines all generally valid relations between Schur indices and exponents. It is known, however, that, for a number of special fields  $E$ , the pairs  $\text{ind}(A), \exp(A)$ ,  $A \in s(E)$ , are subject to much tougher restrictions than those described

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by (1.1) (a). The Brauer  $p$ -dimensions  $\text{Brd}_p(E)$  of  $E$ ,  $p \in \mathbb{P}$ , and their supremum  $\text{Brd}(E)$ , the Brauer dimension of  $E$ , contain essential information about these restrictions. The field  $E$  is said to be of Brauer  $p$ -dimension  $\text{Brd}_p(E) = n$ , where  $n \in \mathbb{Z}$ , if  $n$  is the least integer  $\geq 0$  for which  $\text{ind}(D) \leq \exp(D)^n$  whenever  $D \in d(E)$  and  $[D] \in \text{Br}(E)_p$ ; if no such  $n$  exists, we put  $\text{Brd}_p(E) = \infty$ . In view of (1.1),  $\text{Brd}(E) \leq 1$  if and only if  $\text{ind}(D) = \exp(D)$ , for each  $D \in d(E)$ ;  $\text{Brd}_p(E) = 0$ , for a given  $p$ , if and only if  $\text{Br}(E)_p = \{0\}$ . The absolute Brauer  $p$ -dimension  $\text{abrd}_p(E)$  of  $E$  is defined as the supremum  $\text{Brd}_p(R)$ :  $R \in \text{Fe}(E)$ , where  $\text{Fe}(E)$  is the set of finite extensions of  $E$  in  $E_{\text{sep}}$ . For example, when  $E$  is a global or local field,  $\text{Brd}_p(E) = \text{abrd}_p(E) = 1$ , for all  $p \in \mathbb{P}$ , and there exist  $Y_n \in d(E)$ ,  $n \in \mathbb{N}$ , with  $\text{ind}(Y_n) = n$ , for each  $n$  (see [36], Ch. XII, Sect. 2, and Ch. XIII, Sects. 3 and 6).

The main purpose of this paper is to determine  $\text{Brd}_p(K)$  and to describe index-exponent relations over  $\text{Br}(K)_p$ , provided that  $(K, v)$  is a Henselian (valued) field with a local residue field  $\widehat{K}$ , and  $p \in \mathbb{P}$  is different from  $\text{char}(\widehat{K})$  (for the case of a global field  $\widehat{K}$ , see [10], Sect. 5). Our main result, presented by the following theorem, concerns the case where the value group  $v(K)$  is  $p$ -indivisible, i.e. its quotient group  $v(K)/pv(K)$  is nontrivial:

**Theorem 1.1.** *Assume that  $(K, v)$  is a Henselian field, such that  $\widehat{K}$  is a local field and  $\text{Brd}_p(K) < \infty$ , for some  $p \in \mathbb{P}$  not equal to  $\text{char}(\widehat{K})$ . Let  $\varepsilon_p$  be a primitive  $p$ -th root of unity in  $\widehat{K}_{\text{sep}}$ ,  $\tau(p)$  the dimension of  $v(K)/pv(K)$  as a vector space over the field  $\mathbb{F}_p$  with  $p$  elements,  $m_p = \min\{\tau(p), r_p(\widehat{K})\} > 0$ , and in case  $\varepsilon_p \in \widehat{K}$ , put  $r'_p(\widehat{K}) = r_p(\widehat{K}) - 1$  and  $m'_p = \min\{\tau(p), r'_p(\widehat{K})\}$ . For each  $n \in \mathbb{N}$ , let  $\mu(p, n) = nm_p$ , if  $\varepsilon_p \notin \widehat{K}$ , and  $\mu(p, n) = nm'_p + \nu_n(m_p - m'_p + [(\tau(p) - m_p)/2])$ , if  $\varepsilon_p \in \widehat{K}$  and  $\nu_n = \min\{n, \nu\}$ ,  $\nu$  being the greatest integer for which  $\widehat{K}$  contains a primitive  $p^\nu$ -th root of unity. Then  $\text{Brd}_p(K) = \mu(p, 1)$ ; also, for a pair  $(k, n) \in \mathbb{N}^2$ , there exists  $D_{k,n} \in d(K)$  with  $\text{ind}(D_{k,n}) = p^k$  and  $\exp(D_{k,n}) = p^n$  and only if  $n \leq k \leq \mu(p, n)$ .*

In addition to Theorem 1.1, we find  $\text{Brd}_p(K)$  and describe index-exponent pairs of  $p$ -algebras over  $K$ , provided that  $(K, v)$  is a maximally complete field,  $\text{char}(K) = p$  and  $\widehat{K}$  is a local field. This is obtained in Section 3 as a consequence of a complete description of index-exponent pairs of  $p$ -algebras over maximally complete fields of characteristic  $p$  with perfect residue fields (see Propositions 3.4, 3.5 and Corollary 3.6). The proof of Theorem 1.1 itself is based on the fact that local fields are primarily quasilocal (abbr, PQL), i.e. they are  $p$ -quasilocal fields with respect to every  $p \in \mathbb{P}$ . As a matter of fact, local fields are quasilocal, i.e. their finite extensions are PQL (see [32], Ch. XIII, Sect. 3). When  $E$  is a field with  $r_p(E) > 0$ , for a fixed  $p \in \mathbb{P}$ , we say that  $E$  is  $p$ -quasilocal, if the relative Brauer group  $\text{Br}(E'/E)$  equals the group  ${}_p\text{Br}(E) = \{b \in \text{Br}(E) : pb = 0\}$ , for every degree  $p$  extension  $E'$  of  $E$  in  $E(p)$ . The formula for  $\text{Brd}_p(K)$  given by Theorem 1.1 is deduced from a more general result applying to any  $p$ -quasilocal  $\widehat{K}$  with  $p \neq \text{char}(\widehat{K})$  and  $r_p(\widehat{K}) > 0$ , for some  $p \in \mathbb{P}$ . This result is contained in Theorem 4.1 and its proof relies on the inequality  $\text{Brd}_p(\widehat{K}) \leq 1$ , and on the following relations

between finite extensions of  $\widehat{K}$  in  $\widehat{K}(p)$  and algebras  $\Delta_p \in d(\widehat{K})$  of  $p$ -primary dimensions (see [7], I, Theorems 3.1 and 4.1 (iii)):

- (1.2) (i) A field  $L'_p \in I(\widehat{K}(p)/\widehat{K})$  is embeddable in  $\Delta_p$  as a  $\widehat{K}$ -subalgebra if and only if  $[L'_p : \widehat{K}] \mid \text{ind}(\Delta_p)$ .
- (ii) A finite extension  $L_p$  of  $\widehat{K}$  in  $\widehat{K}(p)$  is a splitting field of  $\Delta_p$ , i.e.  $[\Delta_p] \in \text{Br}(L_p/\widehat{K})$ , if and only if  $[L_p : \widehat{K}] < \infty$  and  $\text{ind}(\Delta_p) \mid [L_p : \widehat{K}]$ .

The description of index-exponent relations over  $\text{Br}(K)_p$ , under the hypotheses of Theorem 1.1, is based on the knowledge of the structure of the (continuous) character group  $C(\widehat{K}(p)/\widehat{K})$  of  $\mathcal{G}(\widehat{K}(p)/\widehat{K})$  as an abstract abelian group (see (6.3) and Remark 6.2). As shown in Sections 5 and 6, this approach leads to a full description of index-exponent relations over  $\text{Br}(K)_p$  whenever  $(K, v)$  is a Henselian field, such that  $\widehat{K}$  is  $p$ -quasilocal and the group  $\mu_p(\widehat{K})$  of roots of unity in  $\widehat{K}$  of  $p$ -primary degrees is nontrivial. The imposed conditions on  $\widehat{K}$  and  $\mu_p(\widehat{K})$  enable one not only to determine the structure of  $C(\widehat{K}(p)/\widehat{K})$  (see (5.1) (a), (6.1) (a), Remark 5.3 and Proposition 5.4). They also make it possible to use it in our proofs in conjunction with the presentability of cyclic  $\widehat{K}$ -algebras of degree  $p$  as symbol algebras, following from Kummer theory (these algebras are defined, for example, in [29], Sect. 15, and in [21], respectively). When  $\text{Br}(\widehat{K})_p \neq \{0\}$ , we rely at crucial points on the fact (see [11], Theorem 3.1) that the canonical correspondence of the set of finite abelian extensions of  $\widehat{K}$  in  $\widehat{K}(p)$  into the set of norm subgroups of  $\widehat{K}^*$  is injective and maps field compositums into group intersections, and field intersections into subgroup products.

The basic notation and terminology used and conventions kept in this paper are standard, like those in [7] and [9]. For a Henselian field  $(K, v)$ ,  $K_{\text{ur}}$  denotes the compositum of inertial extensions of  $K$  in  $K_{\text{sep}}$ ; the notions of an inertial, a nicely semi-ramified (abbr, NSR), an inertially split, and a totally ramified (division)  $K$ -algebra, are defined in [21]. Valuation-theoretic preliminaries used in the sequel are included in Section 2. By a Pythagorean field, we mean a formally real field whose set of squares is additively closed. As usual,  $[r]$  stands for the integral part of any real number  $r \geq 0$ . Given a field extension  $\Lambda/\Psi$ ,  $I(\Lambda/\Psi)$  denotes the set of its intermediate fields. Throughout this paper, Galois groups are viewed as profinite with respect to the Krull topology, and by a profinite group homomorphism, we mean a continuous one. The reader is referred to [26], [16], [21], [29] and [33], for missing definitions concerning field extensions, orderings and valuation theory, simple algebras, Brauer groups and Galois cohomology.

## 2. Preliminaries on Henselian fields and valued extensions

Let  $(K, v)$  be a Krull valued field with a residue field  $\widehat{K}$  and a (totally ordered) value group  $v(K)$ . We say that  $(K, v)$  is Henselian, if  $v$  extends uniquely, up-to an equivalence, to a valuation  $v_L$  on each algebraic extension  $L/K$ . This occurs, for example, if  $(K, v)$  is maximally complete, i.e. it has no valued extension  $(K', v')$ , such that  $K' \neq K$ ,  $\widehat{K}' = \widehat{K}$  and  $v'(K') = v(K)$ .

When  $(K, v)$  is Henselian, we denote by  $\widehat{L}$  the residue field of  $(L, v_L)$  and put  $v(L) = v_L(L)$ , for each algebraic extension  $L/K$ . It is well-known that  $\widehat{L}/\widehat{K}$  is an algebraic extension and  $v(K)$  is a subgroup of  $v(L)$ . Moreover, Ostrowski's theorem states the following (cf. [16], Theorem 17.2.1):

(2.1) If  $L/K$  is finite and  $e(L/K)$  is the index of  $v(K)$  in  $v(L)$ , then  $[\widehat{L} : \widehat{K}]e(L/K)$  divides  $[L : K]$  and  $[L : K][\widehat{L} : \widehat{K}]^{-1}e(L/K)^{-1}$  is not divisible by any  $p \in \mathbb{P}$ ,  $p \neq \text{char}(\widehat{K})$ ; when  $\text{char}(\widehat{K}) \nmid [L : K]$ ,  $[L : K] = [\widehat{L} : \widehat{K}]e(L/K)$ .

It is known (cf. [31], Ch. 2, Sect. 7) that, for any Henselian field  $(K, v)$ , each  $\Delta \in d(K)$  has a unique, up-to an equivalence, valuation  $v_\Delta$  which extends  $v$  and has an abelian value group  $v(\Delta)$ . The group  $v(\Delta)$  is totally ordered and includes  $v(K)$  as an ordered subgroup of index  $e(\Delta/K) \leq [\Delta : K]$ ; the residue division ring  $\widehat{\Delta}$  of  $(\Delta, v_\Delta)$  is a  $\widehat{K}$ -algebra with  $[\widehat{\Delta} : \widehat{K}] \leq [\Delta : K]$ . More precisely, by Ostrowski-Draxl's theorem [14],  $e(\Delta/K)[\widehat{\Delta} : \widehat{K}] \mid [\Delta : K]$ , and if  $\text{char}(\widehat{K}) \nmid \text{ind}(\Delta)$ , then  $[\Delta : K] = e(\Delta/K)[\widehat{\Delta} : \widehat{K}]$ . Note that (2.1) and the Henselity of  $(K, v)$  imply the following:

(2.2) The quotient groups  $v(K)/pv(K)$  and  $v(L)/pv(L)$  are isomorphic, if  $p \in \mathbb{P}$  and  $[L : K] < \infty$ . When  $\text{char}(\widehat{K}) \nmid [L : K]$ , the natural embedding of  $K$  into  $L$  induces canonically an isomorphism  $v(K)/pv(K) \cong v(L)/pv(L)$ .

A finite extension  $R$  of  $K$  is said to be inertial, if  $[R : K] = [\widehat{R} : \widehat{K}]$  and  $\widehat{R}$  is separable over  $\widehat{K}$ . We say that  $R/K$  is totally ramified, if  $[R : K] = e(R/K)$ ;  $R/K$  is called tamely ramified, if  $\widehat{R}/\widehat{K}$  is separable and  $\text{char}(\widehat{K}) \nmid e(R/K)$ . The properties of  $K_{\text{ur}}/K$  used in the sequel are essentially the same as those presented on page 135 of [21], and restated in [9], (3.3). Here we recall some results on central division  $K$ -algebras (most of which can be found in [21]):

(2.3) (a) If  $D \in d(K)$  and  $\text{char}(\widehat{K}) \nmid \text{ind}(D)$ , then  $[D] = [S \otimes_K V \otimes_K T]$ , for some  $S, V, T \in d(K)$ , such that  $S/K$  is inertial,  $V/K$  is NSR,  $T/K$  is totally ramified,  $T \otimes_K K_{\text{ur}} \in d(K_{\text{ur}})$ ,  $\exp(T \otimes_K K_{\text{ur}}) = \exp(T)$ , and  $T$  is a tensor product of totally ramified cyclic  $K$ -algebras (see also [14], Theorem 1);

(b) The set  $\text{IBr}(K)$  of Brauer equivalence classes of inertial  $K$ -algebras  $S' \in d(K)$  is a subgroup of  $\text{Br}(K)$  canonically isomorphic to  $\text{Br}(\widehat{K})$ ;  $\text{Brd}_p(\widehat{K}) \leq \text{Brd}_p(K)$ ,  $p \in \mathbb{P}$ , and equality holds when  $p \neq \text{char}(\widehat{K})$  and  $v(K) = pv(K)$ ;

(c) With assumptions and notation being as in (a), if  $T \neq K$ , then  $K$  contains a primitive root of unity of degree  $\exp(T)$ ; in addition, if  $T_n \in d(K)$  and  $[T_n] = n[T]$ , for some  $n \in \mathbb{N}$ , then  $T_n/K$  is totally ramified;

Statement (2.3) can be supplemented as follows (see, e.g., [10], Sect. 4):

(2.4) If  $D, S, V$  and  $T$  are related as in (2.3) (a), then:

(a)  $\text{IBr}(K)$  contains the class  $n[D]$ , for a given  $n \in \mathbb{N}$ , if and only if  $n$  is divisible by  $\exp(V)$  and  $\exp(T)$ ;

(b)  $D/K$  is inertial if and only if  $V = T = K$ ;  $D/K$  is inertially split, i.e.  $[D] \in \text{Br}(K_{\text{ur}}/K)$ , if and only if  $T = K$ ;

(c)  $\exp(D) = \text{lcm}(\exp(S), \exp(V), \exp(T))$ .

Our next result provides lower and upper bounds on  $\text{Brd}_p(K)$ , under the hypothesis that  $(K, v)$  is a Henselian field with  $\text{Brd}_p(\widehat{K}) < \infty$ , for some  $p \neq \text{char}(\widehat{K})$ . This result can be stated as follows (cf. [10], Theorem 2.3):

**Theorem 2.1.** *Let  $(K, v)$  be a Henselian field with a residue field  $\widehat{K}$  satisfying the condition  $\text{Brd}_p(\widehat{K}) < \infty$ , for some  $p \in \mathbb{P}$  different from  $\text{char}(\widehat{K})$ . Let also  $\tau(p)$  be the dimension of  $v(K)/pv(K)$  over the field  $\mathbb{F}_p$ ,  $\varepsilon_p$  a primitive  $p$ -th root of unity in  $\widehat{K}_{\text{sep}}$ , and  $m_p = \min\{\tau(p), r_p(\widehat{K})\}$ . Then:*

- (a)  $\text{Brd}_p(K) = \infty$  if and only if  $m_p = \infty$  or  $\tau(p) = \infty$  and  $\varepsilon_p \in \widehat{K}$ ;
- (b)  $\max(\text{Brd}_p(\widehat{K}) + [\tau(p)/2], [(\tau(p) + m_p)/2]) \leq \text{Brd}_p(K) \leq \text{Brd}_p(\widehat{K}) + [(\tau(p) + m_p)/2]$ , provided that  $\tau(p) < \infty$  and  $\varepsilon_p \in \widehat{K}$ ;
- (c) When  $m_p < \infty$  and  $\varepsilon_p \notin \widehat{K}$ ,  $m_p \leq \text{Brd}_p(K) \leq \text{Brd}_p(\widehat{K}) + m_p$ .

When  $(K, v)$  is Henselian with  $\text{Brd}_p(\widehat{K}) < \text{Brd}_p(K) = \infty$ , for some  $p \in \mathbb{P}$ ,  $p \neq \text{char}(\widehat{K})$ , index-exponent relations over  $\text{Br}(K)_p$  are fully described by the following consequence of Theorem 2.1, obtained in [10], Sect. 4:

**Corollary 2.2.** *Let  $(K, v)$  be a Henselian field with  $\text{Brd}_p(\widehat{K}) < \infty$  and  $\text{Brd}_p(K) = \infty$ , for some  $p \neq \text{char}(\widehat{K})$ . Then the following alternative holds:*

- (a) For each pair  $(k, n) \in \mathbb{N}^2$  with  $n \leq k$ , there exists  $D_{k,n} \in d(K)$ , such that  $\text{ind}(D_{k,n}) = p^k$  and  $\text{exp}(D_{k,n}) = p^n$ ;
- (b)  $p = 2$  and  $\widehat{K}$  is a Pythagorean field; such being the case, the group  $\text{Br}(K)_2$  has period 2, and there are  $D_m \in d(K)$ ,  $m \in \mathbb{N}$ , with  $\text{ind}(D_m) = 2^m$ .

We conclude these preliminaries with a lemma that is implicitly used in the proofs of the main results of the following Section.

**Lemma 2.3.** *Let  $(K, v)$  be a valued field with  $\text{char}(K) = p > 0$  and  $v(K) \neq pv(K)$ , and let  $\pi$  be an element of  $K^*$  of value  $v(\pi) \notin pv(K)$ . Assume that  $G$  is a finite abelian  $p$ -group of order  $p^t$ . Then there exists a Galois extension  $M$  of  $K$  in  $K(p)$ , such that  $\mathcal{G}(M/K) \cong G$ ,  $v$  is uniquely extendable to a valuation  $v_M$  of  $M$ , up-to an equivalence, and  $v(\pi) \in p^t v_M(M)$ ; in particular,  $v_M(M)/v(K)$  is cyclic and  $(M, v_M)/(K, v)$  is totally ramified.*

*Proof.* First we prove the existence of a sequence  $L'_m, L_m, m \in \mathbb{N}$ , of Galois extensions of  $K$  in  $K(p)$  satisfying the following conditions, for each  $m$ :

- (2.5) (a)  $L'_m/K$  is a  $\mathbb{Z}_p$ -extension, i.e.  $\mathcal{G}(L'_m/K)$  is isomorphic to the additive group  $\mathbb{Z}_p$  of  $p$ -adic integers;  $L_m \in I(L'_m/K)$  and  $[L_m : K] = p$ ;
- (b) The compositums  $M_m = L_1 \dots L_m$  and  $M'_m = L'_1 \dots L'_m$  are Galois extensions of  $K$ , such that  $[M_m : K] = p^m$  and  $\mathcal{G}(M'_m/K) \cong \mathbb{Z}_p^m$ ;
- (c) Every finite extension  $M$  of  $K$  in  $M'_m$  has a unique valuation  $v_M$  extending  $v$ , up-to an equivalence,  $(M, v_M)/(K, v)$  is totally ramified, and  $v(\pi) \in p^t v_M(M)$ , where  $p^t = [M : K]$ .

One may assume without loss of generality that  $v(\pi) < 0$ . Let  $\mathbb{F}$  be the prime subfield of  $K$ ,  $(K_v, \bar{v})$  a Henselization of  $(K, v)$ ,  $\rho(K_v) = \{u^p - u : u \in K_v\}$ ,  $\omega$  the valuation of the field  $\Phi$  induced by  $v$  and for each  $m \in \mathbb{N}$ , let  $L_m$  and  $\Phi_m$  be the root fields in  $K_{\text{sep}}$  over  $K$  and  $\Phi$ , respectively, of the polynomial  $f_m(X) = X^p - X - \pi_m$ , where  $\pi_m = \pi^{1+qm}$ . Identifying  $K_v$  with its  $K$ -isomorphic copy in  $K_{\text{sep}}$ , take a Henselization  $(\Phi_\omega, \bar{\omega})$  of  $(\Phi, \omega)$  as a valued subfield of  $(K_v, \bar{v})$  (this is possible, by [16], Theorem 15.3.5), and denote

by  $\Psi_m = \Phi_1 \dots \Phi_m$  and  $M_m = L_1 \dots L_m$ , for each index  $m$ . It is well-known that  $(K_v, \bar{v})/(K, v)$  and  $(\Phi_\omega, \bar{\omega})/(\Phi, \omega)$  are immediate and  $\rho(K_v)$  is an  $\mathbb{F}$ -subspace of  $K_v$ , and it is easily verified that  $\bar{v}(u') \in pv(K)$  whenever  $u' \in \rho(K_v)$  and  $\bar{v}(u') < 0$ . This implies that the cosets  $\pi_m + \rho(K_v)$ ,  $m \in \mathbb{N}$ , are linearly independent over  $\mathbb{F}$ , so the Artin-Schreier theorem (cf. [26], Ch. VIII, Sect. 6) implies the following, for each  $m \in \mathbb{N}$ :

(2.6)  $L_m/K$ ,  $L_m K_v/K_v$ ,  $\Phi_m/\Phi$  and  $\Phi_m \Phi_\omega/\Phi_\omega$  are degree  $p$  cyclic extensions;  $M_m/K$ ,  $M_m K_v/K_v$ ,  $\Psi_m/\Phi$  and  $\Psi_m \Phi_\omega/\Phi_\omega$  are abelian of degree  $p^m$ .

Note further that, by Witt's lemma (cf. [13], Sect. 15, Lemma 2), for any  $m \in \mathbb{N}$ , there is a  $\mathbb{Z}_p$ -extension  $\Phi'_m$  of  $\Phi$  in  $K_{\text{sep}}$ , such that  $\Phi_m \in I(\Phi'_m/\Phi)$ . Hence, by Galois theory,  $L'_m = \Phi'_m K$  is a  $\mathbb{Z}_p$ -extension of  $K$ . We show that  $M_m$  and the field  $M'_m = L'_1 \dots L'_m$ ,  $m \in \mathbb{N}$ , have the properties required by (2.5). Note first that  $M'_m = \Psi'_m K$ , where  $\Psi'_m = \Phi'_1 \dots \Phi'_m$ . Also, it follows from (2.6) and Galois theory that  $[\Psi_0 \Phi_\omega : \Phi_\omega] = p$  and  $\Psi_0 \in I(\Psi_m/\Phi)$ ; for any degree  $p$  extension  $\Psi_0$  of  $\Phi$  in  $\Psi'_m$ . Hence,  $\Psi_0 \Phi_\omega/\Phi_\omega$  is totally ramified. Let now  $\Psi$  be a finite extension of  $\Phi$  in  $\Psi'_m$ . Observing that  $\widehat{\Phi}$  is a finite field and  $(\Phi_\omega, \bar{\omega})$  is a Henselian discrete valued field, one obtains that each  $\Phi'_\omega \in \text{Fe}(\Phi_\omega)$  is defectless [34], Proposition 2.2, and contains as a subfield an inertial lift of  $\widehat{\Phi}'_\omega$  over  $\Phi_\omega$ . Therefore, Galois theory and our observations on  $\Psi_0$  indicate that  $\Psi \Phi_\omega/\Phi_\omega$  is totally ramified and  $[\Psi K : K] = [\Psi \Phi_\omega : \Phi_\omega] = [\Psi : \Phi]$ . This implies  $\Psi/\Phi$  is totally ramified, which means that  $\Psi/\Phi$  possesses a primitive element  $\theta$  whose minimal polynomial  $f_\theta(X)$  over  $K$  is Eisensteinian relative to  $\omega$  (cf. [18], Ch. 2, (3.6), and [26], Ch. XII, Sects. 2, 3 and 6). Let  $\theta_0$  be the free term of  $f_\theta(X)$ . As  $\pi \in \Phi$ ,  $v(\pi) \notin pv(K)$  and  $\Psi/\Phi$  is a Galois extension, this implies  $\theta$  is a primitive element of  $\Psi K/K$ ,  $p^m w(\theta) = v(\theta_0) = \omega(\theta_0)$  and  $v(\pi) \in p^m w(M_m)$ , for any valuation  $w$  of  $\Psi K$  extending  $v$ . The obtained result proves the uniqueness of  $w$ , up-to an equivalence. It is now easy to see that  $\Psi'_m \cap K_v = \Phi$ , so it follows from Galois theory that the mapping of  $I(\Psi'_m/\Phi)$  into  $I(M'_m/K)$ , by the rule  $\Psi' \rightarrow \Psi' K$ , is bijective with  $\mathcal{G}(\Psi' K/K) \cong \mathcal{G}(\Psi'/\Phi)$ , for each  $\Psi' \in I(\Psi'_m/\Phi)$ . This completes the proof of (2.5) and Lemma 2.3.  $\square$

### 3. Brauer $p$ -dimensions of Henselian fields of characteristic $p$

In this Section we consider index-exponent relations of  $p$ -algebras over Henselian fields of characteristic  $p$ . For this purpose, we need the following lemma whose applicability is guaranteed by Lemma 2.3:

**Lemma 3.1.** *Assume that  $(K, v)$  is a valued field with  $\text{char}(K) = p > 0$  and  $v(K) \neq pv(K)$ ,  $\tau(p)$  is the  $\mathbb{F}_p$ -dimension of  $v(K)/pv(K)$ , and  $L$  is a finite abelian extension of  $K$  in  $K(p)$  satisfying the following conditions:*

- (a)  $[L : K] = p^m$ , the period of  $\mathcal{G}(L/K)$  is equal to  $p^{m'}$ , and  $\mathcal{G}(L/K)$  has a minimal system of  $t$  generators;
- (b)  $L$  has a unique, up-to an equivalence, valuation  $v_L$  extending  $v$ , and the group  $v_L(L)/v(K)$  is cyclic of order  $p^m$ .

Then there exists  $T \in d(K)$ , such that  $\exp(T) = p^{m'}$  and  $T$  possesses a maximal subfield  $K$ -isomorphic to  $L$ , except, possibly, in the case where  $\tau(p) < \infty$  and  $p^{t-\tau(p)} \geq [\hat{K} : \hat{K}^p]$ .

*Proof.* It is clear from Galois theory and the structure of finite abelian groups that  $L = L_1 \dots L_t$  and  $[L : K] = \prod_{j=1}^t [L_j : K]$ , for some cyclic extensions  $L_j/K$ ,  $j = 1, \dots, t$ . Put  $\pi_0 = \pi$  and suppose that there exist elements  $\pi_j \in K^*$ ,  $j = 1, \dots, t$ , and an integer  $\mu$  with  $0 \leq \mu \leq t$ , such that the cosets  $v(\pi_i) + pv(K)$ ,  $i = 0, \dots, \mu$ , are linearly independent over  $\mathbb{F}_p$ , and in case  $\mu < t$ ,  $v(\pi_u) = 0$  and the residue classes  $\hat{\pi}_u$ ,  $u = \mu + 1, \dots, t$ , generate an extension of  $\hat{K}^p$  of degree  $p^{t-\mu}$ . Fix a generator  $\lambda_j$  of  $\mathcal{G}(L_j/K)$ , for  $j = 1, \dots, t$ , denote by  $T$  the  $K$ -algebra  $\otimes_{j=1}^t (L_{j-1}/K, \lambda_{j-1}, \pi_j)$ , where  $\otimes = \otimes_K$ , and put  $T' = T \otimes_K K_v$ . We show that  $T \in d(K)$  (whence  $\exp(T) = \text{per}(\mathcal{G}(L/K))$  and  $\text{ind}(T) = p^m$ ). Clearly, there is a  $K_v$ -isomorphism  $T' \cong \otimes_{j=1}^t (L'_{j-1}/K_v, \lambda'_{j-1}, \pi_j)$ , where  $\otimes = \otimes_{K_v}$  and  $\lambda'_{j-1}$  is the unique  $K_v$ -automorphism of  $L'_{j-1}$  extending  $\lambda_{j-1}$ , for each  $j$ . Therefore, it suffices for the proof of Lemma 3.1 to show that  $T' \in d(K_v)$ . Since, by the proof of Lemma 2.3,  $K_v$  and  $L' = LK_v$  are related as in our lemma, this amounts to proving that  $T \in d(K)$ , for  $(K, v)$  Henselian. Suppose first that  $m = 1$ . As  $L_1/K$  is totally ramified, it follows from the Henselity of  $v$  that  $v(l) \in pv(L_1)$ , for every element  $l$  of the norm group  $N(L_1/K)$ . One also sees that if  $l \in N(L_1/K)$  and  $v_L(l) = 0$ , then  $\hat{K}^p$  contains the residue class  $\hat{l}$ . These observations prove that  $\pi_1 \notin N(L_1/K)$ , so it follows from [29], Sect. 15.1, Proposition b, that  $T_2 \in d(K)$ . Henceforth, we assume that  $m \geq 2$  and view all value groups considered in the rest of the proof as (ordered) subgroups of a fixed divisible hull of  $v(K)$ . Let  $L_0$  be the degree  $p$  extension of  $K$  in  $L_t$ , and  $R_j = L_0 L_j$ ,  $j = 1, \dots, t$ . Put  $\rho_t = \lambda_t^p$ , and in case  $t \geq 2$ , denote by  $\rho_j$  the unique  $L_0$ -automorphism of  $R_j$  extending  $\lambda_j$ , for  $j = 1, \dots, t-1$ . Then the centralizer  $C$  of  $L_0$  in  $T$  is  $L_0$ -isomorphic to  $\otimes_{j=1}^t (R_j/L_0, \rho_j, \pi_j)$ , where  $\otimes = \otimes_{L_0}$ . Therefore, using (2.1) and Lemma 2.3, one easily obtains that it suffices to prove that  $T \in d(K)$  in the case where  $C \in d(L_0)$ .

Denote by  $w$  the valuation of  $C$  extending  $v_{L_0}$ , and by  $\hat{C}$  its residue division ring. It follows from the Ostrowski-Draxl theorem that  $w(C)$  equals the sum of  $v(L)$  and the group generated by  $[L_{i'} : K]^{-1}v(\pi_{i'})$ ,  $i' = 1, \dots, \mu$ . Similarly, it is proved that  $\hat{C}/\hat{K}$  is a purely inseparable field extension unless  $\hat{C} = \hat{K}$ . Moreover, one sees that  $\hat{C} \neq \hat{K}$  if and only if  $\mu < t$ , and when this is the case,  $[\hat{C} : \hat{K}] = \prod_{u=\mu+1}^t [L_u : K]$  and  $\hat{C} = \hat{K}(\eta_{\mu+1}, \dots, \eta_t)$ , where  $\eta_u$  is a root of  $\hat{\pi}_u$  of degree  $[L_u : K]$ , for each index  $u$ . In view of (2.1) and well-known general properties of purely inseparable finite extensions (cf. [26], Ch. VII, Sect. 7), these results show that  $v(\pi_t) \notin pw(C)$ , if  $\mu = t$ , and  $\hat{\pi}_t \notin \hat{C}^p$ , otherwise. Observe now that, by the Skolem-Noether theorem (cf. [29], Sect. 12.6), there exists a  $K$ -isomorphism  $\bar{\rho}_t$  of  $C$  extending  $\lambda_t$ , and it is induced by an inner  $K$ -automorphism of  $T$ . This implies  $w(c) = \bar{\rho}_t(c)$ , for each  $c \in C$ , the products  $c' = \prod_{\kappa=0}^{p-1} \bar{\rho}_t^\kappa(c)$ ,  $c \in C$ , have values  $w(c') \in pw(C)$ , and  $\hat{c}' \in \hat{C}^p$ , if  $w(c) = 0$ . Therefore,  $c' \neq \pi_t$ , for any  $c \in C$ , so it follows from [2], Ch. XI, Theorems 11 and 12, that  $T \in d(K)$ . Let now  $\Lambda$  be the fixed field of  $\mathcal{G}(L/K)^p$ . Then [29], Sect. 15.1, Corollary b, indicates that the class

$p[D] \in \text{Br}(K)$  is represented by a crossed product of  $\Lambda/K$ , defined similarly to  $D$ . Since  $\Lambda/K$  and  $\pi$  are related like  $L/K$  and  $\pi$ , it is now easy to prove, proceeding by induction on  $m'$ , that  $\exp(D) = p^{m'}$ , as claimed.  $\square$

Our next result is of independent interest. It reduces the study of Brauer  $p$ -dimensions of finitely-generated transcendental extensions of a field  $E$  to the special case where  $p \neq \text{char}(E)$  (see [7], for more details).

**Proposition 3.2.** *Let  $E$  be a field with  $\text{char}(E) = p > 0$  and  $[E : E^p] = p^\nu < \infty$ , and  $F/E$  a finitely-generated extension of transcendency degree  $n > 0$ . Then  $n + \nu - 1 \leq \text{Brd}_p(F) \leq \text{abrd}_p(F) \leq n + \nu$ , and when  $n + \nu \geq 2$ ,  $(p^t, p^s) : t, s \in \mathbb{N}, s \leq t \leq (n + \nu - 1)s$ , are index-exponent pairs over  $F$ .*

*Proof.* Our assumptions ensure that  $[F_1 : F_1^p] = p^{n+\nu}$ , for every finite extension  $F_1/F$ , so it follows from [7], Lemma 4.1, and Alberts theory of  $p$ -algebras (cf. [2], Ch. VII, Theorem 28) that  $\text{Brd}_p(F) \leq \text{abrd}_p(F) \leq n + \nu$ . At the same time, it is easy to see that if  $S$  is a subset of  $F$  consisting of  $n$  algebraically independent elements over  $E$ , then any ordering on  $S$  gives rise to a valuation  $v$  of  $F$ , such that  $v(F) = \mathbb{Z}^n$ ,  $v$  induces on  $E$  the trivial valuation, and  $\widehat{F}$  is a finite extension of  $E$ . Therefore,  $[\widehat{F} : \widehat{F}^p] = p^\nu$  (cf. [26], Ch. VII, Sect. 7) and  $v(F)/pv(F)$  is of order  $p^n$ , which enables one to deduce the remaining assertions of Proposition 3.2 from Lemma 3.1.  $\square$

*Remark 3.3.* It is known (see [30], (3.19), or [21], Corollary 6.10) that if  $(K, v)$  is a Henselian field and  $T \in d(K)$  is a tame algebra, in the sense of [30] or [21], then the period  $\text{per}(T/K)$  of the group  $v(T)/v(K)$  divides  $\exp(T)$ . At the same time, by Lemma 3.1 with its proof,  $(K, v)$  can be chosen so that there exist  $T_n \in d(K)$ ,  $n \in \mathbb{N}$ , such that  $\text{ind}(T_n) = \text{per}(T_n/K) = \exp(T_n/K)^n$  and  $[T_n : K] = [\widehat{T}_n : \widehat{K}]e(T_n/K)$ , for each  $n$ .

The following two results fully describe index-exponent pairs of  $p$ -algebras of maximally complete fields of characteristic  $p$  with perfect residue fields.

**Proposition 3.4.** *Let  $(K, v)$  be a valued field of characteristic  $p > 0$ . Suppose that  $v(K)/pv(K)$  is infinite or  $[\widehat{K} : \widehat{K}^p] = \infty$ , where  $\widehat{K}^p = \{\hat{a}^p : \hat{a} \in \widehat{K}\}$ . Then  $(p^k, p^n) : (k, n) \in \mathbb{N}^2, n \leq k$ , are index-exponent pairs over  $K$ .*

*Proof.* Lemma 3.1, [10], Remark 4.3, and our assumptions show that there exist  $D_n \in d(K)$ ,  $n \in \mathbb{N}$ , such that  $\exp(D_n) = p$ ,  $\text{ind}(D_n) = p^n$  and  $D_n$  is a tensor product of degree  $p$  cyclic  $K$ -algebras, for each index  $n$ . Hence, by [8], Lemma 5.2, it is sufficient to prove that  $(p^n, p^n)$ ,  $n \in \mathbb{N}$ , are index-exponent pairs over  $K$ . Note again that, by Witt's lemma, for any cyclic extension  $L$  of  $K$  in  $K(p)$ , we have  $L \in I(L'/K)$ , for some  $\mathbb{Z}_p$ -extension  $L'$  of  $K$  in  $K(p)$ . Let  $\sigma$  be a topological generator of  $\mathcal{G}(L'/K)$ , and for each  $n \in \mathbb{N}$ , let  $L_n$  be the extension of  $K$  in  $L'$  of degree  $p^n$ , and  $\sigma_n$  the automorphism of  $L_n$  induced by  $\sigma$ . Clearly,  $L_n/K$  is cyclic and  $\sigma_n$  is a generator of  $\mathcal{G}(L_n/K)$ . Now choose  $L'$  so that  $(L_1/K, \sigma_1, c) \cong D_1$ , for some  $c \in K^*$ . Then, by [29], Sect. 15.1, Corollary a, the cyclic  $K$ -algebras  $\Delta_n = (L_n/K, \sigma_n, c)$ ,  $n \in \mathbb{N}$ , satisfy  $\text{ind}(\Delta_n) = \exp(\Delta_n) = p^n$ , which completes our proof.  $\square$



**Proposition 3.5.** *Let  $(K, v)$  be a maximally complete field with  $\text{char}(K) = p > 0$  and  $[K : K^p] = p^n$ , for some  $n \in \mathbb{N}$ . Then  $n - 1 \leq \text{Brd}_p(K) \leq n$ . Moreover, if  $\widehat{K}$  is perfect, then:*

- (a)  $\text{Brd}_p(K) = n - 1$  if and only if  $n > r_p(\widehat{K})$ ;
- (b)  $(p^k, p^s) : (k, s) \in \mathbb{N}^2, s \leq k \leq \text{Brd}_p(K)s$ , are index-exponent pairs over  $K$ .
- (c)  $\text{abrd}_p(K) = n - 1$  if and only if the Sylow pro- $p$ -subgroups of the absolute Galois group  $\mathcal{G}_{\widehat{K}}$  are trivial or isomorphic to  $\mathbb{Z}_p$ .

*Proof.* Our assumptions show that  $[K : K^p] = [\widehat{K} : \widehat{K}^p]e(K/K^p)$  (cf. [35], Theorem 31.21), so it follows from Lemma 3.1 and Albert's theory of  $p$ -algebras [2], Ch. VII, Theorem 28, that  $n - 1 \leq \text{Brd}_p(K) \leq n$ , as claimed. In the rest of the proof, we assume that  $\widehat{K}$  is perfect. Suppose first that  $r_p(\widehat{K}) \geq n$ . Then one obtains from Galois theory and Witt's lemma that  $\mathbb{Z}_p^n$  is realizable as a Galois group over  $\widehat{K}$ . Hence, by well-known properties of the natural bijection  $I(K_{\text{ur}}/K) \rightarrow I(\widehat{K}_{\text{sep}}/\widehat{K})$ , there is a Galois extension  $U_n$  of  $K$  in  $K_{\text{ur}}$  with  $\mathcal{G}(U_n/K) \cong \mathbb{Z}_p^n$ . This implies each finite abelian  $p$ -group  $H$  that can be generated by  $n$  elements is isomorphic to  $\mathcal{G}(U_H/K)$ , for some Galois extension  $U_H$  of  $K$  in  $K_{\text{ur}}$ . Observing also that  $v(K)/pv(K)$  has order  $p^n$ , and applying [21], Exercise 4.3, one proves the existence of an NSR-algebra  $N_H \in d(K)$  possessing a maximal subfield  $K$ -isomorphic to  $U_H$ . This result shows that  $\text{Brd}_p(K) = n$ , and reduces the rest of our proof to the special case where  $n > r_p(\widehat{K})$ . Then it follows from [3], Theorem 3.3, and [7], Lemma 4.1, that  $\text{Brd}_p(K) \leq n - 1$ , which completes the proof of Proposition 3.5 (a). The validity of Proposition 3.5 (b) is proved as in the case of  $n \leq \text{Brd}_p(K)$ , using Lemma 3.1 instead of [21], Exercise 4.3. Note finally that  $(L, v_L)$  is maximally complete and  $[L : L^p] = p^n$ , for every  $L \in \text{Fe}(K)$  (cf. [35], Theorem 31.22, and [26], Ch. VII, Sect. 7). In view of Proposition 3.5 (a), this enables one to deduce Proposition 3.5 (c) from [37], Theorem 2, Galois cohomology and Nielsen-Schreier's formula for open subgroups of free pro- $p$ -groups (cf. [33], Ch. I, 4.2, and Ch. II, 2.2).  $\square$

Our next result complements Theorem 1.1 as follows:

**Corollary 3.6.** *Assume that  $(K, v)$  is a maximally complete field,  $\text{char}(K) = p > 0$ ,  $\widehat{K}$  is a local field and  $\tau(p)$  is defined as in Theorem 2.1. Then:*

- (a)  $\text{Brd}_p(K) = \infty$  if and only if  $\tau(p) = \infty$ ; when this holds,  $(p^k, p^n)$  is an index-exponent pair over  $K$ , for each  $(k, n) \in \mathbb{N}^2$  with  $k \geq n$ ;
- (b)  $\text{Brd}_p(K) = \tau(p)$ , provided that  $\tau(p) < \infty$ ; in this case,  $(p^k, p^n)$  is an index-exponent pair over  $K$ , where  $(k, n) \in \mathbb{N}^2$ , if and only if  $n \leq k \leq n\tau(p)$ .

*Proof.* Let  $\omega$  be the natural discrete valuation of  $\widehat{K}$ , and  $\widehat{K}_\omega$  its residue field. It is known (cf. [16], Sect. 5.2) that  $K$  is endowed with a valuation  $w$  (a refinement of  $v$ ), such that  $w(K) = v(K) \oplus \omega(\widehat{K})$ ,  $\omega(\widehat{K})$  is an isolated subgroup of  $w(K)$ ,  $v$  and  $\omega$  are canonically induced by  $w$  and  $\omega(\widehat{K})$  upon  $K$  and  $\widehat{K}$ , respectively, and the residue field  $\widehat{K}_w$  of  $(K, w)$  is isomorphic to  $\widehat{K}_\omega$ . Observing further that, by theorems due to Krull and Hasse-Schmidt-MacLane

(cf. [16], Theorems 12.2.3, 18.4.1, and [35], Theorem 31.24 and page 483),  $(\widehat{K}, \omega)$  is maximally complete and  $(K, w)$  has a maximally complete valued extension  $(K', w')$  with  $\widehat{K}' = \widehat{K}_w$  and  $w'(K') = w(K)$ , one concludes that  $(K', w') = (K, w)$ . Since  $\widehat{K}_w$  is perfect and  $r_p(\widehat{K}_w) = 1$ , this allows one to deduce Corollary 3.6 from Propositions 3.4 and 3.5.  $\square$

When  $(K, v)$  is a Henselian field, such that  $\text{char}(K) = p > 0$ ,  $v(K)$  is a non-Archimedean group,  $v(K)/pv(K)$  is finite and  $[\widehat{K} : \widehat{K}^p] = p^\nu < \infty$ , there is, generally, no formula for  $\text{Brd}_p(K)$  involving only invariants of  $\widehat{K}$  and  $v(K)$ . We illustrate this fact in case  $v(K) = \mathbb{Z}^t$ , for any integer  $t \geq 2$ .

*Example.* Let  $Y_0$  be a field with  $\text{char}(Y_0) = p$  and  $[Y_0 : Y_0^p] = p^\nu$ , and let  $Y_t = Y_0((T_1)) \dots ((T_t))$  be the iterated formal Laurent power series field in  $t$  variables over  $Y_0$ . It is known (see [6], page 2 and further references there) that there exists a sequence  $X_n \in Y_{t-1}$ ,  $n \in \mathbb{N}$ , of algebraically independent elements over the field  $Y_{t-2}(T_{t-1})$ , where  $Y_{t-2} = Y_0((T_1)) \dots ((T_{t-2}))$  in case  $t \geq 3$ . Put  $F_n = Y_{t-2}(T_{t-1}, X_1, \dots, X_n)$ , for each  $n \in \mathbb{N}$ ,  $F_\infty = \bigcup_{n=1}^\infty F_n$ , and  $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ . For any  $n \in \mathbb{N}_\infty$ , denote by  $F'_n$  the separable closure of  $F_n$  in  $Y_{t-1}$ , and by  $v_n$  the valuation of the field  $K_n = F'_n((T_t))$  induced by the natural  $\mathbb{Z}^t$ -valued valuation of  $Y_t$  trivial on  $Y_0$ . It is easily verified that  $(K_n, v_n)$  is Henselian with  $v_n(K_n) = \mathbb{Z}^t$  and  $\widehat{K}_n = Y_0$ , for every index  $n$ . Note also that  $[F'_\infty : F_\infty^{p^p}] = \infty$ , so it follows from Proposition 3.4, applied to the valuation of  $K_n$  induced by the natural discrete valuation of  $Y_t$  trivial on  $Y_{t-1}$ , that  $\text{Brd}_p(K_\infty) = \infty$ . When  $n \in \mathbb{N}$ , we have  $[K_n : K_n^p] = p^{\nu+t+n} = p[F'_n : F_n^{p^p}]$ , which enables one to deduce from Lemma 3.1, [8], Lemma 4.1, and the theory of  $p$ -algebras [2], Ch. VII, Theorem 28 (see also [26], Ch. VII, Sect. 7) that  $\nu + t + n - 1 \leq \text{Brd}_p(K_n) \leq \nu + n + t$ .

#### 4. The Brauer $p$ -dimension of a Henselian field with a $p$ -quasilocal residue field

Let  $(K, v)$  be a Henselian field with a  $p$ -quasilocal field  $\widehat{K}$  and  $r_p(\widehat{K}) > 0$ . Then  $\text{Brd}_p(\widehat{K})_p \leq 1$ , so it follows from Theorem 2.1 that  $\text{Brd}_p(K) = \infty$  if and only if  $m_p = \infty$  or  $\tau(p) = \infty$  and  $\varepsilon_p \in \widehat{K}$ . When  $\text{Brd}_p(K) = \infty$ , index-exponent relations over  $\text{Br}(K)_p$  are described by Corollary 2.2 and the characterization of formally real 2-quasilocal fields, provided by [7], I, Lemma 3.5. When  $\text{Brd}_p(K) < \infty$ ,  $\text{Brd}_p(K)$  is determined as follows:

**Theorem 4.1.** *In the setting of Theorem 2.1, let  $\widehat{K}$  be a  $p$ -quasilocal field,  $m_p > 0$  and  $\text{Brd}_p(K) < \infty$ . Then:*

- (a)  $\text{Brd}_p(K) = u_p$ , where  $u_p = [(\tau(p) + m_p)/2]$ , if  $\varepsilon_p \in \widehat{K}$  and  $\widehat{K}$  is nonreal;  $u_p = m_p$ , if  $\varepsilon_p \notin \widehat{K}$ ;
- (b)  $\text{Br}(K)_2$  is a group of period 2 and  $\text{Brd}_2(K) = 1 + [\tau(2)/2]$ , provided that  $\widehat{K}$  is formally real and  $p = 2$ .

*Proof.* Suppose first that  $\widehat{K}$  is formally real and  $p = 2$ . Then, by [7], I, Lemma 3.5,  $\widehat{K}$  is Pythagorean,  $\widehat{K}(2) = \widehat{K}(\sqrt{-1})$  and  $\text{Br}(\widehat{K})_2$  is a group of order 2. Therefore,  $r_2(\widehat{K}) = 1$  and  $r_2(\widehat{K}(\sqrt{-1})) = 0$ , so it follows from the Merkur'ev-Suslin theorem (see [27], (16.1)), that  $\text{Br}(\widehat{K}(\sqrt{-1}))_2 = \{0\}$ . Note also that  $K$  is Pythagorean, which implies  $2\text{Br}(K) = \{0\}$  (cf. [25], Theorem 3.16, and [15], Theorem 3.1). These observations and [10], Corollary 6.2, prove Theorem 4.1 (b). We turn to the proof of Theorem 4.1 (a), so we assume that  $p > 2$  or  $\widehat{K}$  is a nonreal field. Our argument relies on the following results concerning inertial algebras  $I \in d(K)$  with  $[I] \in \text{Br}(K)_p$ , and inertial extensions  $U$  of  $K$  in  $K(p)$ :

- (4.1) (a)  $\text{ind}(I) = \exp(I)$  and  $I$  is a cyclic  $K$ -algebra;
- (b)  $[I] \in \text{Br}(U/K)$  if and only if  $\text{ind}(U) \mid [U: K]$ ;  $U$  is embeddable in  $I$  as a  $K$ -subalgebra if and only if  $[U: K] \mid \text{ind}(I)$ ;
- (c)  $\text{ind}(I \otimes_K I')$  equals  $\text{ind}(I)$  or  $\text{ind}(I')$ , if  $I' \in d(K)$ ,  $I'/K$  is NSR, and  $[I'] \in \text{Br}(K)_p$ .

Statements (4.1) can be deduced from (1.2), (2.3) (b) and [21], Theorems 3.1 and 5.15. They imply in conjunction with [10], Lemma 4.1, that  $\text{ind}(W) \mid \exp(W)^{m_p}$ , for each  $W \in d(K)$  inertially split over  $K$ . At the same time, it follows from [9], (3.3), and [28], Theorem 1 (see also [21], Exercise 4.3), that there is an NSR-algebra  $W' \in d(K)$  with  $\text{ind}(W') = p^{m_p}$  and  $\exp(W') = p$ . Observe now that, by (2.3) (c),  $d(K)$  consists of inertially split  $K$ -algebras in case  $\varepsilon_p \notin \widehat{K}$  or  $\tau(p) = 1$ . In view of (4.1) and [21], Theorem 4.4 and Lemma 5.14, this yields  $\text{Brd}_p(K) = m_p$ , so it remains for us to prove Theorem 4.1, under the extra hypothesis that  $\varepsilon_p \in \widehat{K}$  and  $\tau(p) \geq 2$ . It is easily obtained from [28], Theorem 1, and [10], Lemmas 4.1 and 4.2, that there exists  $\Delta \in d(K)$  with  $\exp(\Delta) = p$  and  $\text{ind}(\Delta) = p^{\mu(p)}$ , where  $\mu(p) = [(m_p + \tau(p))/2]$ . This means that  $\text{Brd}_p(K) \geq \mu(p)$ , so we have to prove that  $\text{Brd}_p(K) \leq \mu(p)$ . Note first that  $2 \leq m_p \leq r_p(\widehat{K})$ , provided that  $\text{Br}(\widehat{K})_p \neq \{0\}$ . Assuming the opposite and taking into account that  $\varepsilon_p \in \widehat{K}$ , one obtains from the other conditions on  $\widehat{K}$  that it is a nonreal field with  $r_p(\widehat{K}) = 1$ . Hence, by [37], Theorem 2,  $\widehat{K}(p)/\widehat{K}$  is a  $\mathbb{Z}_p$ -extension, i.e.  $\mathcal{G}(\widehat{K}(p)/\widehat{K})$  is isomorphic to the additive group  $\mathbb{Z}_p$  of  $p$ -adic integers. In view of [27], (11.5) and (16.1), and Galois cohomology (cf. [33], Ch. I, 4.2), this requires that  $\text{Br}(\widehat{K})_p = \{0\}$ . As  $\tau(p) \geq 2$ , the obtained contradiction proves the claimed inequalities. Now take an algebra  $D \in d(K)$  so that  $\exp(D) = p^n$ , for some  $n \in \mathbb{N}$ . Suppose that  $S, V$  and  $T \in d(K)$  are related with  $D$  as in (2.3) (a), and fix  $\Theta \in d(K)$  so that  $[\Theta] = [V \otimes_K T]$ . To prove that  $\text{ind}(D) \mid p^{n\mu(p)}$  we need the following statements:

- (4.2) (a) If  $n = 1$ , then  $S, V$  and  $T$  can be chosen so that  $V \otimes_K T = \Theta$ , and  $S = K$  or  $V = K$ .
- (b) If  $n \geq 2$ , then there exists a totally ramified extension  $Y$  of  $K$  in  $K(p)$ , such that  $[Y: K] \mid p^{\mu(p)}$  and either  $\exp(D \otimes_K Y) \mid p^{n-1}$ , or  $\exp(D \otimes_K Y) = \exp(S_Y) = p^n$ ,  $[Y: K]$  divides  $p^{[\tau(p)/2]}$  and  $\exp(V_Y \otimes_Y T_Y)$  divides  $p^{n-1}$ , where  $S_Y, V_Y, T_Y \in d(Y)$  are attached in accordance with (2.3) (a) to the underlying division algebra  $D_Y$  of  $D \otimes_K Y$ .

Statement (4.2) (a) can be deduced from (4.1), [10], (4.8), and well-known properties of cyclic algebras (cf. [29], Sect. 15.1, Proposition b). Since (4.2) (a) implies the assertion of Theorem 4.1 (a) in the case of  $n = 1$ , we assume further that  $n \geq 2$ . The conclusion of (4.2) (b) is obvious, if  $\exp(\Theta) \mid p^{n-1}$  (one may put  $Y = K$ ). Therefore, by (2.4) (c), it suffices to prove (4.2) (b) under the hypothesis that  $\exp(\Theta) = p^n$ . Take  $D_{n-1} \in d(K)$  so that  $[D_{n-1}] = p^{n-1}[D]$  and attach to it a triple  $S_{n-1}, V_{n-1}, T_{n-1} \in d(K)$  in agreement with (4.2) (a). Then  $V_{n-1} \otimes_K T_{n-1}$  contains as a maximal subfield an abelian and totally ramified extension  $Y$  of  $K$ . Identifying  $Y$  with its  $K$ -isomorphic copy in  $K(p)$ , and using (2.4) (a), one sees that it has the properties required by (4.2) (b).

We continue with the proof of Theorem 4.1 (a). For any associative algebra  $B$ , denote by  $Z(B)$  its centre. It is known (cf. [21], Corollary 6.8) that if  $J \in d(K)$  is inertial over  $K$  and  $J' \in d(K)$  is a representative of  $[J \otimes_K \Theta]$ , then  $v(J') = v(\Theta)$ ,  $Z(\hat{J}') = Z(\hat{\Theta})$  and  $[\hat{J}'] = [\hat{J} \otimes_{\hat{K}} \hat{\Theta}] \in \text{Br}(Z(\hat{\Theta}))$ . Note also that the period of the group  $v(J')/v(K)$  divides  $\exp(J')$ , by [30], (3.19) (see also [21], Corollary 6.10). These results imply in conjunction with (4.1) (a), (b) and the Ostrowski-Draxl theorem the following assertions:

(4.3) (a) If  $\exp(\Theta) \mid p^{n-1}$ , then  $\text{ind}(D) \mid p \cdot \text{ind}(S_0 \otimes_K V \otimes_K T)$ , for some  $S_0 \in d(K)$  inertial over  $K$  with  $\exp(S_0) \mid p^{n-1}$ ;

(b) If  $\exp(\Theta) \mid p^{n-1}$  and  $\text{ind}(D) > \text{ind}(I \otimes_K V \otimes_K T)$  whenever  $I \in d(K)$ ,  $I/K$  is inertial and  $\exp(I) \mid p^{n-1}$ , then  $[Z(\hat{D}) : \hat{K}] = p^k$  and  $[\hat{D} : Z(\hat{D})] = p^{2n-2k}$ , for some integer  $k$  with  $0 \leq k < n$ , where  $Z(\hat{D})$  is the centre of  $\hat{D}$ ; in particular,  $\text{ind}(D)^2 \mid p^{2n+(n-1)\tau(p)} \mid p^{nm_p+(n-1)\tau(p)}$ .

Now fix an extension  $Y/K$  and  $Y$ -algebras  $D_Y, S_Y, V_Y, T_Y$  as in (4.2) (b), and take  $\Theta_Y \in d(Y)$  so that  $[\Theta_Y] = [V_Y \otimes_Y T_Y]$ . Arguing by induction on  $n$ , observing that, by (1.1) (b),  $\text{ind}(D) \mid \text{ind}(D_Y)[Y : K]$ , and in case  $\exp(D_Y) = p^n$ , applying (4.3) to  $D_Y, V_Y, T_Y$  and  $\Theta_Y$ , instead of  $D, V, T$  and  $\Theta$ , respectively, one concludes that  $\text{ind}(D)^2 \mid p^{n(m_p+\tau(p))}$ . Thus Theorem 4.1 is proved.  $\square$

*Remark 4.2.* Theorem 4.1 (a) retains its validity, if  $(K, v)$  is a Henselian field with  $\tau(p) < \infty$ ,  $r_p(\hat{K}) = 0$  and  $\mu_p(\hat{K}) \neq \{1\}$ . Then it follows from [27], (16.1), that  $\text{Brd}_p(\hat{K}) = 0$ , so Theorem 2.1 (b) implies  $\text{Brd}_p(K) = [\tau(p)/2]$ .

Our next objective in the present paper is to describe index-exponent relations over  $\text{Br}(K)_p$ , provided that  $(K, v)$  is a Henselian field,  $\hat{K}$  is  $p$ -quasilocal,  $\mu_p(\hat{K}) \neq \{1\}$  and  $\text{Brd}_p(K) < \infty$ , for some  $p \in \mathbb{P}$ . In this Section, we consider only the case where  $\hat{K}$  is formally real and  $p = 2$ . Then  $d(K)$  contains the symbol  $K$ -algebra  $A_{-1}(-1, -1; K)$ , and it follows from [10], Lemma 4.2, that if  $\tau(2) \geq 2$ , then there exist  $D_n \in d(K)$ ,  $n = 1, \dots, [\tau(2)/2]$ , totally ramified over  $K$  with  $\exp(D_n) = 2$  and  $\text{ind}(D_n) = 2^n$ , for each  $n$ . Since  $A_{-1}(-1, -1; K)/K$  is inertial, this implies together with [28], Theorem 1, that  $A_{-1}(-1, -1; K) \otimes_K D_n \in d(K)$  (and  $\text{ind}(A_{-1}(-1, -1; K) \otimes_K D_n) = 2^{n+1}$ ), for  $n = 1, \dots, [\tau(2)/2]$ . In view of (2.3) (b) and Theorem 4.1 (b), these results prove that if  $0 \leq \tau(2) < \infty$ , then  $(1, 1)$  and  $(2^n, 2)$ ,  $n = 1, \dots, 1 + [\tau(2)/2]$ , are all index-exponent pairs over  $\text{Br}(K)_2$ .

### 5. Henselian fields $(K, v)$ with $p$ -quasilocal $\widehat{K}$ satisfying $r_p(\widehat{K}) = \infty$

This Section provides a description of index-exponent relations over  $\text{Br}(K)_p$ , for a Henselian field  $(K, v)$ , such that  $\widehat{K}$  is  $p$ -quasilocal,  $\mu_p(\widehat{K}) \neq \{1\}$  and  $r_p(\widehat{K}) = \infty$ . Our main result concerning this case can be stated as follows:

**Proposition 5.1.** *Under the hypotheses of Theorem 4.1, suppose that  $r_p(\widehat{K}) = \infty$  and  $\varepsilon_p \in \widehat{K}$ . Then:*

- (a) *There exists a sequence  $U_n$ ,  $n \in \mathbb{N}$ , of degree  $p$  extensions of  $K$  in  $K_{\text{ur}}$ , such that  $[U_1 \dots U_n : K] = p^n$  and  $U_n \in I(U'_n/K)$ , where  $U'_n$  is a  $\mathbb{Z}_p$ -extension of  $K$  in  $K_{\text{ur}}$ , for each index  $n$ ;*
- (b) *When  $0 < \tau(p) < \infty$  and  $(n, k) \in \mathbb{N}^2$ ,  $(p^k, p^n)$  is realizable as an index-exponent pair over  $K$  if and only if  $n \leq k \leq \tau(p)n$ .*

*Proof.* (a): The assertion follows at once from Kummer theory, if  $\mu_p(\widehat{K})$  is infinite. We show that it also holds in the special case where  $\text{Br}(\widehat{K})_p = \{0\}$ . Indeed, it follows from [29], Sect. 15.1, Proposition b, that then  $\varepsilon_p$  lies in the norm group  $N(L'/\widehat{K})$ , for every cyclic extension  $L'$  of  $\widehat{K}$  in  $\widehat{K}(p)$ ; hence, by Albert's theorem (cf. [1], Ch. IX, Sect. 6), there is a cyclic extension  $L'_1$  of  $\widehat{K}$  in  $\widehat{K}(p)$ , such that  $L' \in I(L'_1/\widehat{K})$  and  $[L'_1 : L'] = p$ . This observation proves that  $L' \in I(L_1/\widehat{K})$ , for some  $\mathbb{Z}_p$ -extension  $L_1$  of  $\widehat{K}$  in  $\widehat{K}(p)$ . In view of general properties of the natural bijection of  $I(K_{\text{ur}}/K)$  upon  $I(\widehat{K}_{\text{sep}}/\widehat{K})$ , the obtained result shows that each cyclic extension  $U$  of  $K$  in  $K(p) \cap K_{\text{ur}}$  lies in  $I(U'/K)$ , for some  $\mathbb{Z}_p$ -extension  $U'$  of  $K$  in  $K_{\text{ur}}$ . It remains for us to consider the case where  $\text{Br}(\widehat{K})_p \neq \{0\}$  and  $\mu_p(\widehat{K})$  is finite of order  $p^\nu$ . Let  $\delta_\nu$  be a primitive  $p^\nu$ -th root of unity in  $\widehat{K}$ ,  $D(\widehat{K}(p)/\widehat{K})$  the maximal divisible subgroup of  $C(\widehat{K}(p)/\widehat{K})$ , and  $d(p)$  the dimension of  ${}_p\text{Br}(\widehat{K})$  as an  $\mathbb{F}_p$ -vector space. It is known (see, e.g., [22], Ch. 7, Sect. 5) that  $C(\widehat{K}(p)/\widehat{K})$  is an abelian torsion  $p$ -group. Our starting point are the following assertions:

(5.1) (a)  $C(\widehat{K}(p)/\widehat{K})$  is isomorphic to the direct sum  $D(\widehat{K}(p)/\widehat{K}) \oplus \mu_p(\widehat{K})^{d(p)}$ , where  $\mu_p(\widehat{K})^{d(p)}$  is a direct sum of isomorphic copies of  $\mu_p(\widehat{K})$ , indexed by a set of cardinality  $d(p)$ .

(b) A cyclic extension  $M$  of  $\widehat{K}$  in  $\widehat{K}(p)$  lies in  $I(M_\infty/\widehat{K})$ , for some  $\mathbb{Z}_p$ -extension  $M_\infty$  of  $\widehat{K}$  in  $\widehat{K}(p)$  if and only if there is  $M' \in I(\widehat{K}(p)/\widehat{K})$ , such that  $M'/\widehat{K}$  is cyclic,  $M \in I(M'/\widehat{K})$  and  $[M' : M] = p^\nu$ ; this is the case if and only if  $\delta_\nu \in N(M/\widehat{K})$ .

Statement (5.1) (a) is contained in [7], II, Lemma 2.3, the former part of (5.1) (b) is implied by (5.1) (a) and Galois theory, and the latter one follows from Albert's theorem referred to. Let now  $M_\lambda$  be an extension of  $\widehat{K}$  generated by a  $p$ -th root  $\eta_\lambda \in \widehat{K}(p)$  of an element  $\lambda \in \widehat{K}^* \setminus \widehat{K}^{*p}$ . Then  $M_\lambda/\widehat{K}$  is cyclic,  $[M_\lambda : \widehat{K}] = p$  and  $\mathcal{G}(M_\lambda/\widehat{K})$  contains a generator  $\sigma_\lambda$ , such that the cyclic  $\widehat{K}$ -algebra  $(M_\lambda/\widehat{K}, \sigma_\lambda, \delta_\nu)$  is isomorphic to the symbol  $\widehat{K}$ -algebra  $A_{\varepsilon_p}(\lambda, \delta_\nu; \widehat{K})$ . It is well-known that  $A_{\varepsilon_p}(\lambda, \delta_\nu; \widehat{K})$  and  $A_{\varepsilon_p}(\delta_\nu, \lambda; \widehat{K})$  are inversely-isomorphic  $\widehat{K}$ -algebras. Together with [29],

Sect. 15.1, Proposition b, this implies  $\delta_\nu \in N(M_\lambda/\widehat{K})$  if and only if  $\lambda \in N(M_{\delta_\nu}/\widehat{K})$ . Hence, by (5.1) (b), the assertion of Proposition 5.1 (a) is equivalent to the one that  $\widehat{K}^{*p}$  is a subgroup of  $N(M_{\delta_\nu}/\widehat{K})$  of infinite index. Obviously,  $\widehat{K}^{*p} \subseteq N(M_\mu/\widehat{K})$ , for an arbitrary  $\mu \in \widehat{K}^* \setminus \widehat{K}^{*p}$ , so it suffices to show that the group  $N(M_\mu/\widehat{K})/\widehat{K}^{*p}$  is infinite. Fix  $\mu' \in \widehat{K}^* \setminus \widehat{K}^{*p}$  so that  $M_{\mu'} \neq M_\mu$ . Then  $\widehat{K}^*/N(M_{\mu'}/\widehat{K}) \cong {}_p\text{Br}(\widehat{K})$ , by (1.2) and [29], Sect. 15.1, Proposition b, and  $N(M_\mu/\widehat{K})N(M_{\mu'}/\widehat{K}) = \widehat{K}^*$ , by [7], I, Lemma 4.3. Since, by [11], Theorem 3.1,  $N(M_\mu/\widehat{K}) \cap N(M_{\mu'}/\widehat{K}) = N(M_\mu M_{\mu'}/\widehat{K})$ , this yields  $\widehat{K}^*/N(M_{\mu'}/\widehat{K}) \cong N(M_\mu/\widehat{K})/N(M_\mu M_{\mu'}/\widehat{K})$ ,  $\widehat{K}^{*p} \leq N(M_\mu M_{\mu'}/\widehat{K})$  and  $N(M_\mu/\widehat{K})/N(M_\mu M_{\mu'}/\widehat{K}) \cong (N(M_\mu/\widehat{K})/\widehat{K}^{*p})/(N(M_\mu M_{\mu'}/\widehat{K})/\widehat{K}^{*p})$ ; in particular,  ${}_p\text{Br}(\widehat{K})$  is a homomorphic image of  $N(M_\mu/\widehat{K})/\widehat{K}^{*p}$ . Thus it turns out that if  $d(p) = \infty$ , i.e.  ${}_p\text{Br}(\widehat{K})$  is infinite, then  $N(M_\mu/\widehat{K})/\widehat{K}^{*p}$  is infinite as well. Observe now that  $r_p(\widehat{K}) = \infty$  if and only if  $\widehat{K}^*/\widehat{K}^{*p}$  is infinite (cf. [33], Ch. I, 4.1). As the groups  $(\widehat{K}^*/\widehat{K}^{*p})/(N(M_\mu/\widehat{K})/\widehat{K}^{*p})$ ,  $\widehat{K}^*/N(M_\mu/\widehat{K})$  and  ${}_p\text{Br}(\widehat{K})$  are isomorphic, this implies  $N(M_\mu/\widehat{K})/\widehat{K}^{*p}$  is infinite in case  $d(p) < \infty$ , so Proposition 5.1 (a) is proved.

(b): It follows from Proposition 5.1 (a) and Galois theory that, for each finite abelian  $p$ -group  $G$ , there exists a Galois extension  $U_G$  of  $K$  in  $K_{\text{ur}}$  with  $\mathcal{G}(U_G/K) \cong G$ . When  $G$  can be generated by at most  $\tau(p)$  elements, one obtains from [28], Theorem 1, that there is an NSR-algebra  $D_G \in d(K)$  possessing a maximal subfield  $K$ -isomorphic to  $U_G$ . It is therefore clear that there exist  $D_{k,n} \in d(K)$ :  $(k,n) \in \mathbb{N}^2$ ,  $n \leq k \leq \tau(p)n$ , such that  $D_{k,n}/K$  is NSR,  $\text{ind}(D_{k,n}) = p^k$  and  $\exp(D_{k,n}) = p^n$ . This proves Proposition 5.1 (b), since Theorem 4.1 and the condition  $r_p(\widehat{K}) = \infty$  yield  $\text{Brd}_p(K) = \tau(p)$ .  $\square$

It is well-known that Henselian discrete valued fields with quasifinite residue fields are quasilocal (cf. [32], Ch. XIII, Sect. 3). Our next result shows that the conditions of Proposition 5.1 (b) are fulfilled, if  $\text{char}(\widehat{K}) = 0$  and  $\widehat{K}$  possesses a Henselian discrete valuation  $\omega$  with an infinite quasifinite residue field of characteristic  $p$ .

**Proposition 5.2.** *Let  $(E, \omega)$  be a Henselian discrete valued field of zero characteristic with  $\widehat{E}$  quasifinite of characteristic  $p$ . Then:*

- (a)  $r_p(E) = \infty$ , provided that  $\widehat{E}$  is infinite;
- (b)  $C(E(p)/E)$  is divisible if and only if  $\mu_p(E) = \{1\}$ .

*Proof.* (b): Let  $\varepsilon$  be a primitive  $p$ -th root of unity in  $E_{\text{sep}}$ . It is well-known that  $[E(\varepsilon): E] \mid p-1$  (cf. [26], Ch. VIII, Sect. 3). Note also that  $\text{Br}(E') \cong \mathbb{Q}/\mathbb{Z}$ , for every  $E' \in \text{Fe}(E)$ ; in particular, this ensures that the scalar extension map  $\text{Br}(E) \rightarrow \text{Br}(E')$  is surjective. These observations, combined with (1.1) (b) and [29], Sect. 15.1, Proposition b, imply that if  $L$  is a cyclic  $p$ -extension of  $E$  in  $E_{\text{sep}}$ , then  $L(\varepsilon)^* = L^*N(L(\varepsilon)/E(\varepsilon))$ . When  $\varepsilon \notin E$ , this indicates that  $\varepsilon \in N(L(\varepsilon)/E(\varepsilon))$ , which enables one to deduce from [17], Theorem 3, that  $C(E(p)/E)$  is divisible. Suppose now that  $\mu_p(E) \neq \{1\}$  and denote by  $\Gamma_p$  the extension of  $E$  generated by all roots of unity in  $E_{\text{sep}}$  of  $p$ -primary degrees. It is well-known that  $\mathbb{Z}[X]$  contains

the cyclotomic polynomial  $\Phi_{p^n}(X)$  of order  $p^n$  (and degree  $p^{n-1}(p-1)$ ), and the polynomial  $\Phi_{p^n}(X+1)$  is Eisensteinian over  $\mathbb{Z}$  relative to  $p$ . This implies that  $p^{n-1}(p-1)\omega_{\Gamma_p}(\varepsilon_n) = \omega(p)$ , where  $\varepsilon_n \in \Gamma_p$  is a primitive  $p^n$ -th root of unity. As  $\omega$  is discrete and  $\omega(p) \neq 0$ , the noted equalities prove that  $\mu_p(E)$ . In view of (5.1) (a) and the nontriviality of  $\text{Br}(E)_p$ , the obtained result ensures that  $C(E(p)/E) \neq pC(E(p)/E)$ .

(a): Assume that  $\widehat{E}$  is infinite, fix a uniform element  $\pi \in E$  and take elements  $a_n \in E$ ,  $n \in \mathbb{N}$ , so that  $\omega(a_n) = 0$  and the residue classes  $\hat{a}_n$ ,  $n \in \mathbb{N}$ , are linearly independent over the prime subfield of  $\widehat{E}$ . It is easily verified that the cosets  $(1 + a_n\pi)E^{*p}$ ,  $n \in \mathbb{N}$ , are linearly independent over  $\mathbb{F}_p$ . This means that  $E^{*p}$  is a subgroup of  $E^*$  of infinite index. At the same time, it is clear from local class field theory that if  $L_j$ ,  $j = 1, \dots, n$ , are cyclic extensions of  $E$  in  $E(p)$  of degree  $p$ , then  $E^{*p}$  is included in  $N(L_1 \dots L_n/E)$ , which in turn is a subgroup of  $E^*$  of index equal to  $[L_1 \dots L_n : E]$ . Finally, it follows from the quasilocal property of  $E$  that if  $a \in E^* \setminus E^{*p}$ ,  $D \in d(E)$  and  $\text{ind}(D) = p$ , then there exists a cyclic extension  $Y$  of  $E$  in  $E(p)$ , such that  $[Y : E] = p$  and  $D \cong (Y/E, \tau, a)$ , for some generator  $\tau$  of  $\mathcal{G}(Y/E)$ . Hence, by [29], Sect. 15.1, Proposition b,  $a \notin N(Y/E)$ , which means that  $E^{*p}$  equals the intersection of the norm groups of cyclic degree  $p$  extensions of  $E$ . Now the equality  $r_p(E) = \infty$  becomes obvious, so Proposition 5.2 is proved.  $\square$

*Remark 5.3.* Assume that  $(K, v)$  is Henselian field with  $p$ -quasilocal  $\widehat{K}$  and  $\mu_p(\widehat{K}) \neq \{1\}$ . Then it follows from [7], II, Lemma 2.3, that  $C(\widehat{K}(p)/\widehat{K})$  is divisible if and only if  $\text{Br}(\widehat{K})_p = \{0\}$  or  $\mu_p(\widehat{K})$  is infinite. When this holds, one obtains by the method of proving Proposition 5.1 (b) that if  $0 < \text{Brd}_p(K) < \infty$  and  $(k, n) \in \mathbb{N}^2$ , then  $(p^k, p^n)$  is an index-exponent pair over  $K$  if and only if  $n \leq k \leq \text{Brd}_p(K)n$ . Conversely, it is well-known that, for any divisible abelian torsion  $p$ -group  $\Pi$ , there exists a field  $E_\Pi$ , such that  $\mu_p(E_\Pi) \neq \{1\}$ ,  $\text{Br}(E_\Pi)_p = \{0\}$  and  $C(E_\Pi(p)/E_\Pi) \cong \Pi$ .

It is worth noting in connection with (5.1) (a) that the  $\mathbb{F}_p$ -dimension  $d(p)$  of  ${}_p\text{Br}(E)$  is perhaps the most important invariant of a  $p$ -quasilocal field  $E$  with  $r_p(E) > 0$ . This is illustrated, e.g., by [19], Theorem 23.1, and [7], I, Theorem 3.1 and Lemma 3.5, which show that  $d(p)$  fully determines the structure of  $\text{Br}(E)_p$ . Also, it follows from [11], Theorem 3.1, that if  $d(p) > 0$ , then for each finite extension  $M$  of  $E$  in  $E(p)$ ,  $E^*/N(M/E)$  is isomorphic to the direct sum  $\mathcal{G}(M/E)^{d(p)}$  of isomorphic copies of  $\mathcal{G}(M/E)$ , taken over a set of cardinality  $d(p)$ . When  $d(p) = 0$ , we have  $E^* = N(R/E)$ , for all  $R \in I(E(p)/E) \cap \text{Fe}(E)$  (cf. [7], I, Lemma 4.2 (ii)). These results attract interest in the fact that each divisible abelian torsion  $p$ -group  $T_p$  is isomorphic to  $\text{Br}(E(T_p))_p$ , for some  $p$ -quasilocal field  $E(T_p)$ . In view of [7], I, Theorem 3.1 and Lemma 3.5, this property of  $T_p$  can be obtained as a consequence of the following result (see [12], Theorem 1.2 and Proposition 6.4):

(5.2) An abelian torsion group  $T$  is isomorphic to  $\text{Br}(E(T))$ , for some PQL-field  $E(T)$  if and only if it satisfies one of the following two conditions:

(a)  $T$  is divisible; when this holds,  $E(T)$  is necessarily nonreal. Moreover, for a given field  $E_0$ ,  $E(T)$  can be defined so as to be a quasilocal field and an

extension of  $E_0$ , such that  $E_0$  is algebraically closed in  $E(T)$  and the scalar extension map  $\text{Br}(E(T)) \rightarrow \text{Br}(\Lambda)$  is surjective, for each  $\Lambda \in \text{Fe}(E(T))$ ;

(b) The  $p$ -components  $T_p$  are divisible, for every  $p \in \mathbb{P} \setminus \{2\}$ , and the group  $T_2$  is of order 2; such being the case,  $E(T)$  is formally real.

Statement (5.2) is a refinement of [4], Theorem 3.9, which in turn generalizes [20], Example 2.1 (cf. also [4], Theorem 3.8, [5], Theorem 4, and [12], Theorem 1.2 (i), for more details). When  $T$  is divisible,  $E_0$  is a field of at most countable cardinality  $d(0)$ , and  $t$  is an infinite cardinal number such that  $t \geq d(p)$ , for all  $p \in \mathbb{P} \cup \{0\}$ , the quasilocal field  $E(T)$  in (5.2) (a) can be chosen among those extensions of  $E_0$  of transcendency degree  $t$ , which satisfy  $r_p(E(T)) = t$ ,  $p \in \mathbb{P}$  (see [12], Remark 5.4). At the same time, the condition that  $E_0$  is algebraically closed in  $E$  ensures that  $\mu_p(E) = \mu_p(E_0)$ , for each  $p \in \mathbb{P}$ . In addition, it is a well-known consequence of Galois theory and the irreducibility of cyclotomic polynomials over the field  $\mathbb{Q}$  of rational numbers that every subgroup  $\Gamma$  of  $\mathbb{Q}/\mathbb{Z}$  is isomorphic to the group  $\mu(\Phi_\Gamma)$  of roots of unity in some algebraic extension  $\Phi_\Gamma$  of  $\mathbb{Q}$ . Therefore, applying (5.2) (a) to the case of  $T = T_p$ , for a given  $p \in \mathbb{P}$ , and using (5.1) (a) as well as the structure and the injectivity of divisible abelian torsion  $p$ -groups (cf. [19], Theorems 23.1 and 24.5), one proves the following assertion:

**Proposition 5.4.** *Let  $W$  be an abelian torsion  $p$ -group, for some  $p \in \mathbb{P}$ , and let  $D(W)$  be the maximal divisible subgroup of  $W$ . Suppose that  $W$  contains infinitely many elements of order  $p$ . Then there is a  $p$ -quasilocal field  $F_W$  with  $\mu_p(F_W) \neq \{1\}$  and  $C(F_W(p)/F_W) \cong W$ , if and only if,  $W/D(W)$  is embeddable as a subgroup in  $D(W)$ , and in case  $W \neq D(W)$ , it decomposes into the direct sum of cyclic groups of order  $p^n$ , for some  $n \in \mathbb{N}$ .*

## 6. Proof of Theorem 1.1

Our first result completes the description of index-exponent relations over  $\text{Br}(K)_p$ , for a Henselian field  $(K, v)$  with a  $p$ -quasilocal  $\widehat{K}$  and  $\mu_p(\widehat{K}) \neq \{1\}$ .

**Proposition 6.1.** *With assumptions and notation being as in Theorem 4.1, let  $\text{Brd}_p(\widehat{K}) \neq 0$ ,  $\varepsilon_p \in \widehat{K}$ ,  $\mu_p(\widehat{K})$  be a finite group of order  $p^\nu$ ,  $2 \leq r_p(\widehat{K}) = r < \infty$ ,  $r' = r - 1$ ,  $m' = \min\{\tau(p), r'\}$ , and for each  $n \in \mathbb{N}$ , let  $\nu_n = \min\{n, \nu\}$  and  $\mu(p, n) = nm' + \nu_n(m_p - m' + [(\tau(p) - m_p)/2])$ . If  $(k, n) \in \mathbb{N}^2$ , then  $(p^k, p^n)$  is an index-exponent pair over  $K$  if and only if  $n \leq k \leq \mu(p, n)$ .*

*Proof.* First we prove the following assertions:

(6.1) (a)  $C(\widehat{K}(p)/\widehat{K}) \cong \mathbb{Z}(p^\infty)^{r'} \oplus \mu_p(\widehat{K})$  and  $\mathcal{G}(\widehat{K}(p)_{\text{ab}}/\widehat{K}) \cong \mathbb{Z}_p^{r'} \times \mu_p(\widehat{K})$ , where  $\mathbb{Z}(p^\infty)$  is the quasicyclic  $p$ -group and  $\widehat{K}(p)_{\text{ab}}$  is the compositum of finite abelian extensions of  $\widehat{K}$  in  $\widehat{K}(p)$ ;

(b) Statement (5.1) (b) retains validity in the setting of Proposition 6.1.

The inequality  $2 \leq r$  and the  $p$ -quasilocality of  $\widehat{K}$  ensure that  $\widehat{K}$  is nonreal and  $\text{Br}(\widehat{K})_p$  is divisible (cf. [7], I, Theorem 3.1 and Lemma 3.5). As  $\varepsilon_p \in \widehat{K}$  and  $r < \infty$ , they also imply  $\mathcal{G}(\widehat{K}(p)/\widehat{K})$  is a Demushkin group, in the



sense of [24] and [33], and  $\text{Br}(\widehat{K})_p \cong \mathbb{Z}(p^\infty)$  (see [11], Proposition 5.1 and Corollary 5.3). Therefore, (6.1) (a) can be deduced from [7], II, Lemma 2.3, and general properties of the natural bijection  $I(K_{\text{ur}}/K) \rightarrow I(\widehat{K}_{\text{sep}}/\widehat{K})$ . As to (6.1) (b), it follows from (6.1) (a) and Albert's theorem.

We continue with the proof of Proposition 5.1. Statement (2.3) (b), the isomorphism  $\text{Br}(\widehat{K})_p \cong \mathbb{Z}(p^\infty)$ , and the equality  $\text{Brd}_p(\widehat{K}) = 1$  imply that  $(p^m, p^m)$ ,  $m \in \mathbb{N}$ , are index-exponent pairs over both  $\widehat{K}$  and  $K$ . In view of Theorem 4.1, this proves Proposition 6.1 in the special case where  $\tau(p) = 1$ . Henceforth, we assume that  $\tau(p) \geq 2$ . Suppose first that  $n \in \mathbb{N}$  and  $n \leq \nu$ . Then, by Theorem 4.1,  $\text{ind}(\Delta_n) \mid p^{\mu(p,n)}$ , for each  $\Delta_n \in d(K)$  with  $\exp(\Delta_n) = p^n$ . Using [28], Theorem 1, and the natural bijection between  $I(Y/K)$  and the set of subgroups of  $v(Y)/v(K)$ , for any finite abelian tamely and totally ramified extension  $Y/K$  (cf. [31], Ch. 3, Theorem 2), one obtains that, for each  $k \in \mathbb{N}$  with  $n \leq k \leq \mu(p,n)$ , there exist an NSR-algebra  $V_{n,k} \in d(K)$  and a totally ramified  $T_{n,k} \in d(K)$ , such that  $V_{n,k} \otimes_K T_{n,k} \in d(K)$ ,  $\exp(V_{n,k} \otimes_K T_{n,k}) = p^n$  and  $\text{ind}(V_{n,k} \otimes_K T_{n,k}) = p^k$ . These observations and the former part of (1.1) (a) prove Proposition 6.1 when  $n \leq \nu$ . The rest of the proof is carried out by induction on  $n \geq \nu$ . The basis of the induction is provided by Theorem 4.1, which allows us to assume that  $n > \nu$  and  $\text{ind}(X) \mid p^{\mu(p,n-1)}$  whenever  $X \in d(K)$  and  $\exp(X) \mid p^{n-1}$ . Fix an algebra  $D \in d(K)$  so that  $\exp(D) = p^n$  and attach to  $D$  algebras  $S, V, T \in d(K)$  as in (2.3) (a). Clearly, if  $\exp(V) \mid p^{n-1}$ , then  $\exp(V \otimes_K T) \mid p^{n-1}$ , so (4.3) and the inductive hypothesis imply  $\text{ind}(D) \mid p^{1+\mu(p,n-1)} \mid p^{\mu(p,n)}$ , as claimed. In view of (2.4), it remains to consider the case where  $\exp(V) = p^n$ . Let  $\Sigma, D_\nu \in d(K)$  satisfy  $[\Sigma] = [S \otimes_K V]$  and  $[D_\nu] = p^\nu[D] (= p^\nu[\Sigma])$ . Then, by (2.4) (c),  $\exp(\Sigma) = p^n$ , and it follows from (4.1) and [29], Sect. 15.1, Corollary b and Proposition b, that  $\Sigma/K$  is NSR. Also,  $\exp(D_\nu) \mid p^{n-\nu}$ , and (2.3) (c) and [29], Sect. 15.1, Corollary b, imply  $D_\nu/K$  is NSR. In particular,  $D_\nu$  contains as a maximal subfield an inertial extension  $U_\nu$  of  $K$ , and by [21], Theorem 4.4,  $U_\nu/K$  is abelian and  $\mathcal{G}(U_\nu/K)$  has a system of  $\tau(p)$  generators. Moreover, it follows from (6.1), Galois theory and [29], Sect. 15.1, Corollary b, that  $U_\nu$  has a  $K$ -isomorphic copy from  $I(U'_\nu/K)$ , for the Galois extension  $U'_\nu$  of  $K$  in  $K_{\text{ur}}$  with  $\mathcal{G}(U'_\nu/K) \cong \mathbb{Z}_p^{r'}$ . Therefore,  $\mathcal{G}(U_\nu/K)$  has a system of  $r'$  generators, so [21], Theorem 4.4 (or [10], Lemma 4.1), leads to the following conclusion:

(6.2)  $\text{ind}(D_\nu) \mid p^{(n-\nu)m'}$  and  $D_\nu$  contains as a maximal subfield a  $K$ -isomorphic copy of a totally ramified extension  $\Phi_\nu$  of  $K$  in  $K(p)$ .

Statement (6.2) shows that  $[D_\nu] \in \text{Br}(\Phi_\nu/K)$ ,  $[\Phi_\nu: K] = \text{ind}(D_\nu)$  and  $\widehat{\Phi}_\nu = \widehat{K}$ . Hence,  $\exp(D \otimes_K \Phi_\nu) \mid p^\nu$  and  $r_p(\widehat{\Phi}_\nu) = r_p(\widehat{K})$ , so it follows from (2.2) and Theorem 4.1 that  $\text{ind}(D \otimes_K \Phi_\nu) \mid p^{\nu\mu(p)}$ , where  $\mu(p) = [(m_p + \tau(p))/2]$ . As  $\mu(p,n) = (n - \nu)m' + \nu\mu_p$ , it is now easy to see that  $\text{ind}(D) \mid p^{\mu(p,n)}$ , as required. Suppose finally that  $(k,n) \in \mathbb{N}^2$  and  $n \leq k \leq \mu(p,n)$ . Then [21], Exercise 4.3, [28], Theorem 1, the above-noted properties of  $U'_\nu$ , and those of intermediate fields of an abelian tamely and totally ramified finite extension of  $K$ , imply the existence of  $D_{k,n} \in d(K)$  with  $\text{ind}(D_{k,n}) = p^k$  and  $\exp(D_{k,n}) = p^n$ . Moreover, one can ensure that  $D_{k,n} \cong N_{k,n} \otimes_K D'_{k,n}$ , for

some  $N_{k,n}, D'_{k,n} \in d(K)$ , such that  $N_{k,n}$  is NSR and  $D'_{k,n}$  is totally ramified over  $K$ . Proposition 6.1 is proved.  $\square$

We are now in a position to prove Theorem 1.1. As noted in Section 1,  $\widehat{K}$  is quasilocal, and by assumption, it is complete with respect to a discrete valuation  $\omega$  whose residue field  $\widehat{K}_\omega$  is finite. This implies  $(\widehat{K}, \omega)$  is Henselian,  $\mu_p(\widehat{K})$  is finite, and in case  $p \neq \text{char}(\widehat{K}_\omega)$ ,  $\varepsilon_p \in \widehat{K}$  if and only if  $p$  divides the order  $o(\widehat{K}_\omega^*)$  of  $\widehat{K}_\omega^*$ . Put  $r = r_p(\widehat{K})$ , and denote by  $\widehat{K}(p)_{\text{ab}}$  the compositum of abelian finite extensions of  $\widehat{K}$  in  $\widehat{K}(p)$ . It is known (see [23], Sect. 10.1 and Theorem 10.5) that if  $\varepsilon_p \notin \widehat{K}$ , then

(6.3) (a)  $\mathcal{G}(\widehat{K}(p)/\widehat{K}) \cong \mathbb{Z}_p$ , provided that  $p \neq \text{char}(\widehat{K}_\omega)$ ;

(b) When  $\text{char}(\widehat{K}) = 0$  and  $\text{char}(\widehat{K}_\omega) = p$ ,  $\mathcal{G}(\widehat{K}(p)/\widehat{K})$  is a free pro- $p$ -group, and  $\mathcal{G}(\widehat{K}(p)_{\text{ab}}/\widehat{K}) \cong \mathbb{Z}_p^r$ ; in addition,  $\widehat{K}$  is a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers and  $r = [\widehat{K} : \mathbb{Q}_p] + 1$ .

Note further that, by Theorem 4.1,  $\text{Brd}_p(K) = m_p$ , and by (2.3) (c), every  $D \in d(K)$  is inertially split over  $K$ . These results enable one to deduce the assertion of Theorem 1.1 (in case  $\varepsilon_p \notin \widehat{K}$ ) from (6.3), [21], Exercise 4.3, and [28], Theorem 1, by the method of proving Proposition 5.1 (b).

Let now  $\varepsilon_p \in \widehat{K}$ . Then Theorem 4.1 yields  $\text{Brd}_p(K) = \mu(p, 1)$ , and Proposition 6.1 implies that if  $(k, n) \in \mathbb{N}^2$ , then  $(p^k, p^n)$  is an index-exponent pair over  $K$  if and only if  $n \leq k \leq \mu(p, n)$ . This completes our proof.

*Remark 6.2.* In the setting of Theorem 1.1, with its proof, if  $\varepsilon_p \in \widehat{K}$ , then  $r = r_p(\widehat{K})$  is determined as follows: (i)  $r = 2$ , if  $p \neq \text{char}(\widehat{K}_\omega)$ ; (ii) when  $p = \text{char}(\widehat{K}_\omega)$  and  $\text{char}(\widehat{K}) = 0$ ,  $\widehat{K}/\mathbb{Q}_p$  is a finite extension and  $r = [\widehat{K} : \mathbb{Q}_p] + 2$  (see [23], Sect. 10.1, and [24], Sect. 5). For a  $p$ -quasilocal field  $E$  with  $\text{Br}(E)_p \neq \{0\}$ ,  $\mu_p(E) \neq \{1\}$  and  $3 \leq r_p(E) < \infty$ , it is an open problem whether there exists a local field  $L_E$ , such that  $\mathcal{G}(L_E(p)/L_E) \cong \mathcal{G}(E(p)/E)$ .

**Corollary 6.3.** Assume that  $(K, v)$  is a Henselian field, such that  $\widehat{K}$  is a local field, and let  $\omega$  be the usual discrete valuation of  $\widehat{K}$ . Denote by  $\widehat{K}_\omega$  the residue field of  $(\widehat{K}, \omega)$ , and suppose that  $\tau(p)$  is defined as in Theorem 1.1, for each  $p \in \mathbb{P}$ ,  $p \neq \text{char}(\widehat{K})$ . Then  $\text{abrd}_p(K) = 1 + \lceil \tau(p)/2 \rceil$ , provided that  $p \neq \text{char}(\widehat{K}_\omega)$ ;  $\text{abrd}_p(K) = \max\{1, \tau(p)\}$ , if  $\text{char}(\widehat{K}) = 0$  and  $\text{char}(\widehat{K}_\omega) = p$ .

*Proof.* In view of (1.1) (b) and [9], (3.3), it suffices to consider only the special case where  $\mu_p(\widehat{K}) \neq \{1\}$ . Then our conclusion follows from Remark 6.2 and the fact that  $\widehat{K}(p)/\widehat{K}$  is an infinite extension.  $\square$

*Remark 6.4.* Note that the conclusions of Theorem 1.1 hold, if the assumption on  $\widehat{K}$  is replaced by the milder one that  $\widehat{K}$  has a Henselian discrete valuation  $\omega$  with a quasifinite residue field  $\widehat{K}_\omega$ . In the first place, then  $\widehat{K}$  is quasilocal and Theorem 4.1 (a) applies to every  $p \in \mathbb{P}$ ,  $p \neq \text{char}(K)$ . Secondly, it is known (e.g., [18], Ch. 2, (3.5)) that if  $p \neq \text{char}(\widehat{K}_\omega)$ , then

$r_p(\widehat{K}) \leq 2$ , and equality holds if and only if  $\mu_p(\widehat{K}) \neq \{1\}$ . Moreover,  $\mathcal{G}(\widehat{K}_{\text{ab}}(p)/\widehat{K}) \cong \mathbb{Z}_p \times \mu_p(\widehat{K})$ , provided that  $\mu_p(\widehat{K}) < \infty$ ;  $\mathcal{G}(\widehat{K}_{\text{ab}}(p)/\widehat{K}) \cong \mathbb{Z}_p^2$ , otherwise. Thereby, index-exponent relations over  $\text{Br}(K)_p$  are described, in the former case, as in Theorem 1.1, and in the latter one, by Remark 5.3. Finally, if  $\text{char}(\widehat{K}) = 0$  and  $p = \text{char}(\widehat{K}_\omega)$ , then it follows from Theorem 4.1 (a) and Proposition 5.2 that  $\text{Brd}_p(K) \leq \tau(p)$ , and equality holds in case  $\widehat{K}_\omega$  is infinite. In addition, using (5.1) (a) and Proposition 5.2, one obtains as in the proof of Proposition 5.1 (b) that  $(p^k, p^n): k, n \in \mathbb{N}, n \leq k \leq n\text{Brd}_p(K)$ , are index-exponent pairs over  $K$  unless  $r_p(\widehat{K}) \leq \tau(p)$  and  $\mu_p(\widehat{K}) \neq \{1\}$ . The same holds, by the proof of Corollary 3.6, if  $(K, v)$  is maximally complete and  $\text{char}(K) = p > 0$ .

Assuming that  $(K, v)$  is a Henselian field and  $\widehat{K}$  is a local field, summing-up (1.1) (a), Theorem 1.1, Corollary 2.2, and the latter part of (2.3) (b), and using the equalities  $\text{Brd}_p(\widehat{K}) = 1$ ,  $p \in \mathbb{P}$ , together with (6.3) and Remark 6.2, one obtains a complete description of the restrictions on index-exponent pairs over  $K$  not divisible by  $\text{char}(\widehat{K})$ . In view of Remark 6.4, the divisibility restriction can be removed, if  $(K, v)$  is maximally complete.

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INSTITUTE OF MATHEMATICS AND INFORMATICS, BULGARIAN ACADEMY OF SCIENCES,  
1113 SOFIA, BULGARIA

E-mail address: chipchak@math.bas.bg